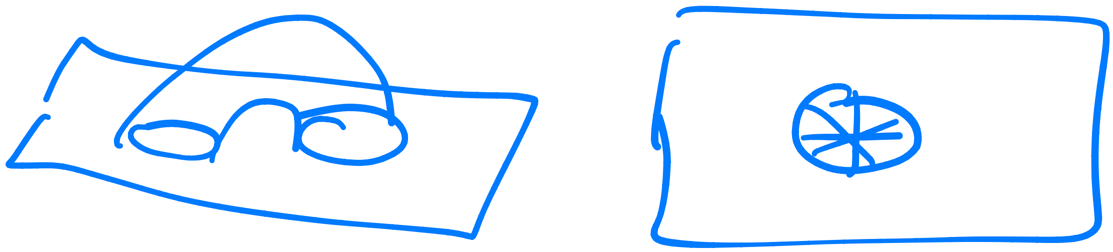
Lecture 2 - Critical Graphs III

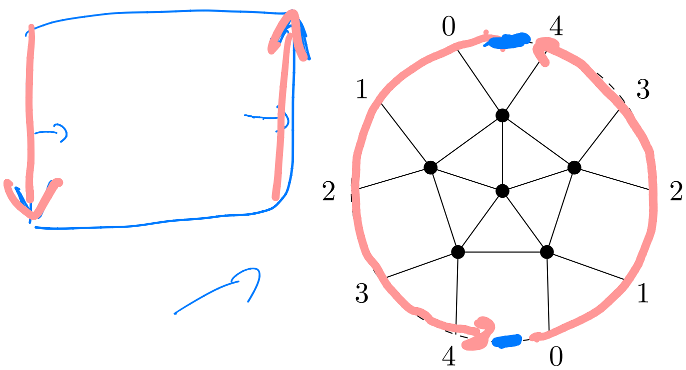
1 Coloring graphs on surfaces

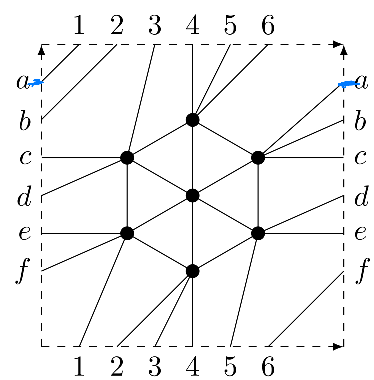
Euler genus of a surface $\mathcal{S}$ is $2\times$ the number of handles plus the number of cross-caps. Every graph G embedded in $\mathcal{S}$ satisfies $n - m + f \geq 2 - g$, where n, m, f, g count vertices, edges, faces, Eulerian genus, respecitvely.

1: What is the Euler genus of the plane, projective plane, torus, double torus and the Klein bottle?



2: Verify the Euler's formula for the two following graphs.





PROJECTIVE PLANE

K_6

K_7

TORUS

Theorem 1 (Heawood, 1890)**.** *Let G be embedded on a surface $\mathcal{S}$ of Euler genus $g > 0$. Then,*

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 24g}}{2} \right\rfloor = H(g).$$

$g \geq 0$ ← HARD

7

$H(1) \geq 6$ $H(0) = 4$

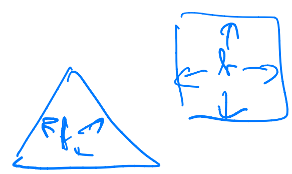
Proof. Let G is a critical k-chromatic graph on a surface $\mathcal{S}$. Then $n \geq k \geq 6$.

3: Count the number of edges using $\delta(G)$, Eulers's formula and trick that each face has at least 3 edges.

$n - m + f \geq 2 - g$ $3f \leq 2m$

$2m \geq (k-1)n$

$3n - \cancel{3}m + \cancel{3f} \geq 6 - 3g$



f

g

$2m \geq (k-1)n$ $3n - 6m + 3g \geq 6 - 3g$ $\to$ $3n - 6 + 3g \geq m$

$6n - 12 + 6g \geq (k-1)n$ $n \geq k$

$0 \geq (k-7)n + 12 - 6g \geq (k-7)k + 12 - 6g$ □

$k_{1,2} = \dfrac{7 \pm \sqrt{49 - 4(12-6g)}}{2}$ $k^2 - 7k + 12 - 6g$

It turned that this bound is sharp by embedding the complete graph $K_{H(g)}$ on $\mathcal{S}_g$ with the only one exception for the Klein bottle. Showing this takes over 100 pages proof with special technics of "current graphs" developed. And, regarding the Klein bottle K_7 is not embeddable on this surface, and it has been shown the graphs on this surface has chromatic number at most 6. The Heawood bound also holds for the sphere by the Four Color Theorem. This all together is nowadays called The Map Color Theorem.

Recall

Theorem 2 (Dirac)**.** *Let $G \neq K_k$ be a k-critical graph on n vertices and with m edges. Then*

$$2m \geq (k-1)n + k - 3. \tag{1}$$

Using the inequality (1), Dirac gave a refinement of The Map Color Theorem.

Theorem 3 (Dirac)**.** *Let G be embedded on a surface $\mathcal{S}$ of Euler genus $g = 2$ or $g \geq 4$. Then, $\chi(G) \leq H(g) - 1$ unless G contains $K_{H(g)}$ as a subgraph.*

Proof. **4:** Try to prove the theorem in a similar way as the Heawood. Note: maybe one needs to try with computer(s) some polynomials later. Goal: Start with an h-chromatic graph on n vertices, where $h = H(g)$ and end with a polynomial containing only g and h that contradicts $H(g)$.

G IS h-CHROMATIC IN S K_h IS NOT IN G

$2m \geq (h-1)n + h - 3$ DIRAC

$m \leq 3n + 3g - 6$ EULER

$(h-7)n + h - 3 \leq 6g - 12$ $\leftarrow$ $(h+2) \leq n$

$(h-7)(h+2) + h - 3 \leq 6g - 12$

FOR SMALL g ASSUME $g \geq 12$

$h \leq \dfrac{4 - \sqrt{36 + 24g}}{2}$, $h = H(g) \geq \left\lfloor \dfrac{\cancel{5}^{\,7} + \sqrt{1+24g}}{2} \right\rfloor$

CONTRADICTION FOR $g \geq 12$

□

The above theorem also hods for the cases $g = 1, 3$ but for them we need different methods to deal with. Sometimes this refinement of the Map Color Theorem is called the Dirac Map Color Theorem.

We also show that for $k \geq 7$ the number of k-critical graphs on a surface is finite. $K = 6$ MORE DIFFICULT

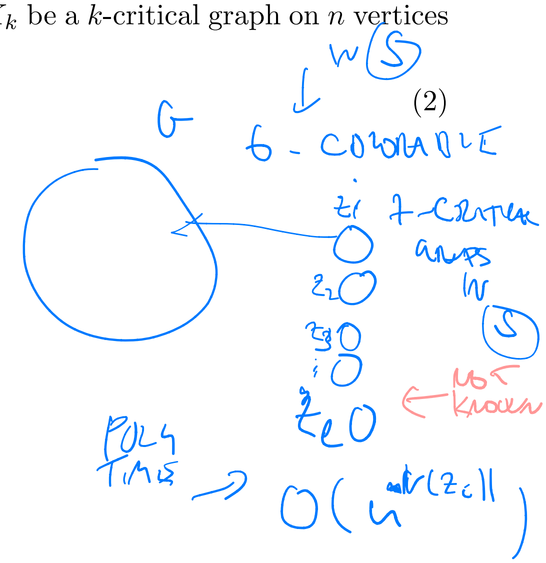
Proposition 4. *Let $\mathcal{S}$ be a surface and $k \geq 7$. Then the number of k-critical graphs that embeds on $\mathcal{S}$ is finite.*

Proof. **5:** Prove the proposition using the equation from Gallai. Let $G \neq K_k$ be a k-critical graph on n vertices and with m edges. Then

$$2m \geq (k-1)n + \frac{k-3}{k^2-3}n + \frac{2(k-1)}{k^2-3}. \tag{2}$$

Goal is to obtain an upper bound on n.

COMBINE WITH EULER'S FORMULA



$m \leq 3n + 3g - 6$ EULER

$2m \leq 6n + 6g - 12$

$6n + 6g - 12 \geq (k-1) \cdot n + \frac{k-3}{k^2-3} n + \frac{2(k-1)}{k^2-3}$

$$\frac{6g - 12 - \frac{2(k-1)}{k^2-3}}{(k-7) + \frac{k-3}{k^2-3}} \geq n$$

□

Thomassen showed that for any (orientable and non-orientable) surface, the number of 6-critical graphs is finite. This is best possible, since for every $3 \leq k \leq 5$, there are infinitely many k-critical graphs, that can be embedded on any fixed surface except the plane.